Math 254-1 Exam 1 Solutions

1. Carefully state the definition of “spanning”. Give two examples for $\mathbb{R}^2$.

   A set of vectors S is spanning if every vector in the vector space can be achieved through linear combinations of S. Equivalently, S is spanning if span(S) is the whole vector space. Many examples are possible. Any basis, such as $\{(1,0),(0,1)\}$, will work. But other examples are possible too, such as $\{(1,1),(1,2),(1,3)\}$.

2. Let $u = [1\ 2\ 3]$, and $v = [0\ 7\ 15]$. For each of the following, determine what *type* they are (undefined, scalar, matrix/vector). If a matrix/vector, specify the dimensions. **DO NOT CALCULATE ANY NUMBERS.**

   (a) uvu (b) uv^Tu (c) u^Tvu^T (d) $(u \cdot v) \times u$ (e) $(u \times v) \cdot u$

   u, v are 1×3; u^T, v^T are 3×1. Hence uvu has pattern $(1 \times 3)(1 \times 3)(1 \times 3)$; neither matrix multiplication is possible, hence (a) is undefined. uv^Tu has $(1\times 3)(3\times 1)(1\times 3)$; both matrix multiplications are possible, and the result of (b) is a 1×3 matrix (or a row 3-vector). u^Tvu^T has $(3 \times 1)(1 \times 3)(3 \times 1)$; both matrix multiplications are possible, and the result of (c) is a 3×1 matrix (or a column 3-vector). $u \cdot v$ gives a scalar, hence (d) is undefined since cross product requires two vectors. (e) is a scalar, because $(u \times v)$ is a 3-vector, hence its dot product with u can be calculated and is a scalar.

3. Let $u = (1, 2, 3)$, and $v = (15, -7, 0)$. Are these vectors orthogonal? Be sure to justify your answer.

   We calculate $u \cdot v = 1(15) + 2(-7) + 3(0) = 1$. Since this is nonzero, these vectors are not orthogonal.

4. For $A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 0 & -1 \\ 1 & 5 \end{bmatrix}$, calculate AB and BA.

   $AB = \begin{bmatrix} 0+0-1 & 0-1-5 \\ -2+0+3 & -1+0+15 \end{bmatrix} = \begin{bmatrix} -1 & -6 \\ 1 & 14 \end{bmatrix}$. $BA = \begin{bmatrix} 0-1 & 2+0 & -2+3 \\ 0+1 & 0+0 & 0-3 \\ 0-5 & 1+0 & -1+15 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 0 & -3 \\ -5 & 1 & 14 \end{bmatrix}$.

5. For $u = (1, 0, 2)$ and $v = (0, -3, 1)$, calculate $u \times v$ and $v \times u$.

   Method 1, determinant formula: $u \times v = \begin{vmatrix} 0 & 2 \\ -3 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 0 \\ 0 & -3 \end{vmatrix} \mathbf{k} =$
   $= (0 + 6)\mathbf{i} - (1 - 0)\mathbf{j} + (-3 + 0)\mathbf{k} = 6\mathbf{i} - 1\mathbf{j} - 3\mathbf{k} = (6, -1, -3)$
   $v \times u = \begin{vmatrix} -3 & 1 \\ 0 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 0 & -3 \\ 1 & 0 \end{vmatrix} \mathbf{k} = (-6 + 0)\mathbf{i} - (0 - 1)\mathbf{j} + (0 + 3)\mathbf{k} =$
   $= -6\mathbf{i} + 1\mathbf{j} + 3\mathbf{k} = (-6, 1, 3)$
   Method 2, $\mathbf{i}, \mathbf{j}, \mathbf{k}$ technique: $u \times v = (\mathbf{i} + 2\mathbf{k}) \times (-3\mathbf{j} + \mathbf{k}) = -3(\mathbf{i} \times \mathbf{j}) + (\mathbf{i} \times \mathbf{k}) - 6(\mathbf{k} \times \mathbf{j}) + 2(\mathbf{k} \times \mathbf{k}) = -3\mathbf{k} + (-\mathbf{j}) - 6(-\mathbf{i}) + 2(0) = -3\mathbf{k} - \mathbf{j} + 6\mathbf{i} = (6, -1, -3)$
   $v \times u = (-3\mathbf{j} + \mathbf{k}) \times (\mathbf{i} + 2\mathbf{k}) = -3(\mathbf{j} \times \mathbf{i}) - 6(\mathbf{j} \times \mathbf{k}) + (\mathbf{k} \times \mathbf{i}) + 2(\mathbf{k} \times \mathbf{k}) = -3(-\mathbf{k}) - 6(\mathbf{i}) + (\mathbf{j}) + 2(0) = 3\mathbf{k} - 6\mathbf{i} + \mathbf{j} = (-6, 1, 3)$